

Inverse problem of electroseismic conversion. I: Inversion of Maxwell's equations with internal data

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Abstract

Pride (1994, Phys. Rev. B 50 1567896) derived the governing model of electroseismic conversion, in which Maxwell's equations are coupled with Biot's equations through an electrokinetic mobility parameter. The inverse problem of electroseismic conversion was first studied by Chen and Yang (2013, Inverse Problem 29 115006). By following the construction of Complex Geometrical Optics (CGO) solutions to a matrix Schrödinger equation introduced by Ola and Somersalo (1996, SIAM J. Appl. Math. 56 No. 4 1129-1145), we analyze the reconstruction of conductivity, permittivity and the electrokinetic mobility parameter in Maxwell's equations with internal measurements, while allowing the magnetic permeability μ to be a variable function. We show that knowledge of two internal data sets associated with well-chosen boundary electric sources uniquely determines these parameters. Moreover, a Lipschitz-type stability is obtained based on the same set.

1 Introduction

In fluid-saturated porous media, an electrical double layer (EDL) is formed at the interface (the pore boundaries) of the fluid and solid rock. The fluid side of the EDL is charged with positive ions and the solid side with negative electrons. When electric or magnetic fields impinge on the EDL, the electrokinetic phenomenon causes movement of the fluid relative to rock frame and thus emits seismic waves, which can be remotely detected. This effect is named electroseismic conversion. Conversely, a seismic wave can cause the separation of charges and therefore generate electromagnetic fields. At the intersection of two formation layers, the singularities in electrical, hydraulic and mechanical properties will lead to a discontinuity in the

induced electromagnetic fields, and emit electro-magnetic waves, which can be remotely observed. This effect is named seismoelectric conversion. In fact, these two ways of conversion always happen simultaneously.

The wave propagation in fluid-saturated porous media was studied by Biot [3, 4]. In Biot's theory, in addition to the conventional compressional and shear waves, a compressional slow wave appears. The first experimental observation of this slow wave was obtained by Plona [14]. The predication of the slow wave has been quantitatively confirmed by Pride [15]. Based on Biot's theory, Pride [15] also developed the governing equations of electroseismic conversion, which are

$$\nabla \wedge E = i\omega\mu H, \quad (1)$$

$$\nabla \wedge H = (\sigma - i\epsilon\omega)E + L(-\nabla p + \omega^2\rho_f u) + J_s, \quad (2)$$

$$-\omega^2(\rho u + \rho_f w) = \nabla \cdot \tau, \quad (3)$$

$$-i\omega w = LE + \frac{\kappa}{\eta}(-\nabla p + \omega^2\rho_f u), \quad (4)$$

$$\tau = (\lambda\nabla \cdot u + C\nabla \cdot w)I + G(\nabla u + \nabla u^T), \quad (5)$$

$$-p = C\nabla \cdot u + M\nabla \cdot w, \quad (6)$$

where the first two are Maxwell's equations, and the remaining ones are Biot's equations. The notation is as follows:

E electric field,

H magnetizing field or magnetic field intensity,

ω seismic wave frequency,

σ conductivity,

ϵ dielectric constant or relative permittivity,

μ magnetic permeability,

J_s source current,

p pore pressure,

ρ_f density of pore fluid,

L electro-kinetic mobility parameter,

κ fluid flow permeability,

u solid displacement,

w relative fluid displacement,

τ bulk stress tensor,

η viscosity of pore fluid,

λ, G Lamé parameters of elasticity,

C, M Biot moduli parameters.

Some basic properties of the coupling effect were studied by Pride and Haartsen in [16].

Under the assumption that the coupling is weak, depending on the source, we focus on one way coupling and ignore the second way which is weaker as it is caused by induced waves. In the case of a seismic boundary source generating waves, seismoelectric conversion will dominate and the governing equations are given as above but with the term LE removed from (4). Thompson and Gist [18] made the first field measurements in 1993 demonstrating the application of seismoelectric conversion as a survey tool. Zhu *et al.* [24, 25] performed a series of laboratory experiments in model wells and studied the application of seismoelectric conversion as a bore-hole logging tool as well as a cross-hole logging tool. In 2011, the laboratory experiments by Schakel [17] also indicated the promising application of seismoelectric conversion for subsurface exploration.

On the other hand, starting with electric field boundary sources, electroseismic conversion is dominant and seismoelectric conversion is negligible, in which case the governing model is as above but with the term $L(-\nabla p + \omega^2 \rho_f u)$ removed in (2). In 2005, White [21] established a forward model for the electroseismic method by combining Maxwell's equations and the elastic wave equation, while the initial amplitude of elastic waves is calculated according to Prides equations with a high-frequency asymptotic theory. In 2007, Thompson *et al.* at ExxonMobil [19] presented results from field tests of electroseismic conversion in Texas and Canada. By using specially designed boundary electric current waveforms at dominant frequencies of 8Hz, 18Hz and 25Hz, their tests over gas sands and carbonate oil reservoirs succeeded in delineating known hydrocarbon accumulations from depths up to 1500m, which suggested applicability of electroseismic conversion at significant depths. The laboratory experiments by Schakel [17] demonstrated that higher frequencies in electric sources lead to smaller aptitude in induced seismic waves. The critical advantage of exploiting electroseismic conversion is that the experiments provide certain internal data for the inverse problem for the (time-harmonic) Maxwell's equations and, as we will show, lead to well-posedness of this problem. In this paper, we analyze the inverse problem of electroseismic conversion, which was first studied mathematically by Chen and Yang [7], with internal data given by $\Sigma := LE$. The analysis of recovering these internal data from boundary measurements using Biot's equations will be presented in a separate paper.

Let Ω be an open bounded subset of \mathbb{R}^3 with smooth boundary $\partial\Omega$. Let $D = -i\nabla$. The time-harmonic Maxwell's equations are given by

$$\begin{cases} D \wedge H + \omega\gamma E = 0, \\ D \wedge E - \omega\mu H = 0, \end{cases} \quad (7)$$

where $\omega > 0$ is a fixed angular frequency and $\gamma = \epsilon + i\sigma/\omega$. We assume that $\mu, \gamma, \epsilon, \sigma \in H^s(\mathbb{R}^3)$, $s > 3/2$, satisfy

$$\mu > 0 \text{ and } \gamma \neq 0 \text{ in } \mathbb{R}^3. \quad (8)$$

By substitution, we have the curl-curl form of Maxwell's equations

$$D \wedge \mu^{-1} D \wedge E + \omega^2 \gamma E = 0. \quad (9)$$

The boundary source is expressed in terms of the boundary tangential components of the electric field,

$$G := tE \quad \text{on } \partial\Omega, \quad (10)$$

where tE is the tangent component of E . We also assume that the internal data of the form

$$\Sigma := LE \quad \text{in } \Omega$$

are given. We define the forward operator Λ_G by,

$$(L, \gamma) \rightarrow \Sigma = \Lambda_G(L, \gamma).$$

The problem studied in this paper is the inversion of operator Λ_G . Precisely, given properly chosen the boundary values of the electrical field, G , can we recover the coupling coefficient L and the complex parameter γ from the internal data Σ ?

Inverse problems of Maxwell's equations with other types of internal data are also studied in [9, 2]. One reconstruction methodology of the inverse problem with internal data, inspired by Bal and Uhlmann's study of photo-acoustic tomography (PAT) [1], primarily consists of converting the governing equation to a transport equation and constructing Complex Geometrical Optics (CGO) solutions to the governing equations. The unique and stable solvability of the transport equation relies on the estimate of the vector field in the transport equation associated with the explicitly constructed CGO solutions. The same methodology was followed by Chan and Yang [7], converting Maxwell's equations to a transport equation while constructing CGO solutions to Maxwell equations. They proved the uniqueness and stability of the reconstruction of the coupling coefficient L and conductivity σ in the case of a constant magnetic permeability μ . Here, we consider the general case with variable μ including the recovery of relative permittivity.

Our construction of CGO solutions for variable μ employs the idea of converting Maxwell's equations to a matrix Schrödinger equation. The matrix Schrödinger equation formulation was first developed by Ola and Somersalo [13] to study the inverse boundary value problem in electromagnetics; the relevant analysis was simplified in [12].

This paper is organized as follows: in Section 2, we introduce the matrix Schrödinger equation the CGO solutions of which are constructed in Section 3. Our main theorems are stated and proven in Section 4, while we derive the transport equation and prove the uniqueness result in Section 4.1. We prove the stability results in Sections 4.2 and 4.3. In Section 5, we address the temporal behavior of CGO solutions.

2 Matrix Schrödinger equation

If $\mu \equiv \mu_0$ is constant, the curl-curl form of Maxwell's equation in (9) is simply given by

$$D \wedge D \wedge E + \omega^2 \gamma \mu_0 E = 0, \quad (11)$$

the CGO solutions of which were constructed by Colton [8] and extended to high order Sobolev spaces by Chen and Yang [7]. However, their construction method fails for non-constant μ . In the present work, we convert the first-order system of Maxwell's equations to a matrix Schrödinger equation and construct corresponding CGO solutions following [12, 5].

Let $\mu, \epsilon, \sigma \in H^s(\mathbb{R}^3)$, $s > \frac{3}{2}$, satisfy (8). Taking the divergence of (7) gives

$$\begin{cases} D \cdot (\gamma E) = 0, \\ D \cdot (\mu H) = 0. \end{cases} \quad (12)$$

Let $\alpha = \log \gamma$ and $\beta = \log \mu$. By combining (7) and (12), we get

$$\begin{cases} D \cdot E + D\alpha \cdot E = 0, \\ -D \wedge E + \omega \mu H = 0, \\ D \cdot H + D\beta \cdot H = 0, \\ D \wedge H + \omega \gamma E = 0. \end{cases} \quad (13)$$

The above system contains 8 equations and 6 unknown components in E and H . We write (13) as a 8×8 matrix system,

$$\left[\begin{pmatrix} * & 0 & * & D \cdot \\ * & 0 & * & -D \wedge \\ * & D \cdot & * & 0 \\ * & D \wedge & * & 0 \end{pmatrix} + \begin{pmatrix} * & 0 & * & D\alpha \cdot \\ * & \omega \mu I_3 & * & 0 \\ * & D\beta \cdot & * & 0 \\ * & 0 & * & \omega \gamma I_3 \end{pmatrix} \right] \begin{pmatrix} 0 \\ H \\ 0 \\ E \end{pmatrix} = 0. \quad (14)$$

We define

$$P_+(D) = \begin{pmatrix} 0 & D \cdot \\ D & D \wedge \end{pmatrix}, \quad P_-(D) = \begin{pmatrix} 0 & D \cdot \\ D & -D \wedge \end{pmatrix},$$

which satisfy

$$\begin{aligned} P_+(D)P_-(D) &= P_-(D)P_+(D) = -\Delta I_4, \\ P_+(D)^* &= P_-(D), \quad P_-(D)^* = P_+(D). \end{aligned}$$

We also define

$$P^\mp(D) = \begin{pmatrix} 0 & P_-(D) \\ P_+(D) & 0 \end{pmatrix}.$$

Then $P^\mp(D)P^\mp(D) = -\Delta I_8$ and $P^\mp(D)^* = P^\mp(D)$. We denote

$$P^\mp(a, b) = \begin{pmatrix} 0 & P_-(b) \\ P_+(a) & 0 \end{pmatrix}, \quad P^\pm(a, b) = \begin{pmatrix} 0 & P_+(b) \\ P_-(a) & 0 \end{pmatrix},$$

where $a, b \in \mathbb{C}^3$, and denote

$$\text{diag}(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Here A, B are 4×4 diagonal complex-valued matrices.

We now introduce scalar fields Φ and Ψ , and $X = (\Phi, H, \Psi, E)^t$, so that the matrix system in (14) attains the form

$$(P^\mp(D) + V_{\mu, \gamma})X = 0 \quad \text{in } \Omega,$$

where

$$V_{\mu, \gamma} = \begin{pmatrix} \omega \mu & 0 & 0 & D\alpha \cdot \\ 0 & \omega \mu I_3 & D\alpha & 0 \\ 0 & D\beta \cdot & \omega \gamma & 0 \\ D\beta & 0 & 0 & \omega \gamma I_3 \end{pmatrix}.$$

Solutions to this system with $\Phi = \Psi = 0$ correspond to solutions of the original Maxwell system. By changing variables so that

$$Y = \text{diag}(\mu^{1/2}, \gamma^{1/2})X, \quad (15)$$

we have

$$(P^\mp(D) + V_{\mu,\gamma})X = 0 \quad \Leftrightarrow \quad (P^\mp(D) + W_{\mu,\gamma})Y = 0, \quad (16)$$

where $\kappa = \omega(\gamma\mu)^{1/2}$ and

$$W_{\mu,\gamma} = \begin{pmatrix} \kappa I_4 & \frac{1}{2}P_+(D\alpha) \\ \frac{1}{2}P_-(D\beta) & \kappa I_4 \end{pmatrix} = \kappa I_8 + \frac{1}{2}P^\pm(D\beta, D\alpha).$$

A direct calculation leads to the following

Lemma 2.1 ([5]). *One has*

$$\begin{aligned} (P^\mp(D) + W_{\mu,\gamma})(P^\mp(D) - W_{\mu,\gamma}^t) &= -\Delta I_8 + \tilde{Q}_{\mu,\gamma}, \\ (P^\mp(D) - W_{\mu,\gamma}^t)(P^\mp(D) + W_{\mu,\gamma}) &= -\Delta I_8 + \tilde{Q}'_{\mu,\gamma}. \end{aligned}$$

Here, the matrix potentials, $\tilde{Q}_{\mu,\gamma}$ and $\tilde{Q}'_{\mu,\gamma}$, are given by

$$\begin{aligned} \tilde{Q}_{\mu,\gamma} &= \frac{1}{2} \left(\begin{array}{cc|cc} \Delta\alpha & 0 & & \\ 0 & 2\nabla\nabla\alpha - \Delta\alpha I_3 & & 0 \\ \hline & 0 & \Delta\beta & 0 \\ & & 0 & 2\nabla\nabla\beta - \Delta\beta I_3 \end{array} \right) \\ &\quad - \left(\begin{array}{cc|cc} (\kappa^2 + \frac{1}{4}(D\alpha)^2)I_4 & & 0 & D\kappa \\ & & 2D\kappa & 0 \\ \hline 0 & 0 & & \\ 0 & -2D\kappa\wedge & (\kappa^2 + \frac{1}{4}(D\beta)^2)I_4 & \end{array} \right), \end{aligned}$$

and

$$\begin{aligned} \tilde{Q}'_{\mu,\gamma} &= \frac{1}{2} \left(\begin{array}{cc|cc} \Delta\beta & 0 & & \\ 0 & 2\nabla\nabla\beta - \Delta\beta I_3 & & 0 \\ \hline & 0 & \Delta\alpha & 0 \\ & & 0 & 2\nabla\nabla\alpha - \Delta\alpha I_3 \end{array} \right) \\ &\quad - \left(\begin{array}{cc|cc} (\kappa^2 + \frac{1}{4}(D\beta)^2)I_4 & & 0 & D\kappa \\ & & 2D\kappa & 0 \\ \hline 0 & 0 & & \\ 0 & -2D\kappa\wedge & (\kappa^2 + \frac{1}{4}(D\alpha)^2)I_4 & \end{array} \right), \end{aligned}$$

with $\nabla\nabla f = (\partial_{x_j, x_k}^2 f)_{j,k=1}^3$.

We can extend μ and γ to \mathbb{R}^3 so that for some nonzero constants μ_0 and γ_0 , $\mu - \mu_0$ and $\gamma - \gamma_0$ are compactly supported. We also assume that Z is the solution to

$$(P^\mp(D) - W_{\mu,\gamma}^t)Z = Y. \quad (17)$$

Lemma 2.1 then implies that Z solves the matrix Schrödinger equation

$$-(\Delta + k^2)I_8 + Q_{\mu,\gamma}Z = 0, \quad (18)$$

where $k = \omega(\mu_0\gamma_0)^{1/2}$ and $Q_{\mu,\gamma} = k^2 I_8 + \tilde{Q}_{\mu,\gamma}$. It follows that $Q_{\mu,\gamma}$ is compactly supported.

In the following section, we will construct solutions to (18) and thus solutions to Maxwell's equations according to (15) and (17).

3 Complex Geometrical Optics(CGO) solutions

Sylvester and Uhlmann [20] constructed CGO solutions to the scalar Schrödinger equation. Ola and Somersalo [12] followed the Sylvester-Uhlmann method to construct CGO solutions to the matrix Schrödinger equation given in (18). The CGO solutions by Ola and Somersalo are in a weighted L^2 space. In the present work, we apply the Sylvester-Uhlmann method to construct CGO solutions in higher order Sobolev spaces.

We first introduce some notation. Let the space L_δ^2 for $\delta \in \mathbb{R}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{L_\delta^2}$ defined by

$$\|u\|_{L_\delta^2} = \left(\int_{\mathbb{R}^3} \langle x \rangle^{2\delta} |u|^2 dx \right)^{1/2}, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

We also define the space H_δ^s for $s > 0$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{H_\delta^s}$ defined by

$$\|u\|_{H_\delta^s} = \left(\int_{\mathbb{R}^3} \langle x \rangle^{2\delta} |(I - \Delta)^{\frac{s}{2}} u|^2 dx \right)^{1/2}.$$

Here $(I - \Delta)^{\frac{s}{2}} u$ is defined as the inverse Fourier transform of $\langle \xi \rangle^s \hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$. When $\delta = 0$, this is the standard Sobolev space $H^s(\mathbb{R}^3)$ of order s .

It is proven in [20] that for $|\zeta| \geq c > 0$ and $v \in L_{\delta+1}^2$ with $-1 < \delta < 0$, equation

$$(\Delta - 2\zeta \cdot D)u = v, \tag{19}$$

admits a unique weak solution $u \in L_\delta^2$ with

$$\|u\|_{L_\delta^2} \leq C \frac{\|v\|_{L_{\delta+1}^2}}{|\zeta|}, \tag{20}$$

for some constant $C = C(\delta, c)$. We note that $(\Delta + 2i\zeta \cdot D)$ and $(I - \Delta)^s$ are constant coefficient operators and hence commute. We deduce that for $v \in H_{\delta+1}^s$, for $s \geq 0$, (19) admits a unique solution $u \in H_\delta^s$ with

$$\|u\|_{H_\delta^s} \leq C(\delta, c) \frac{\|v\|_{H_{\delta+1}^s}}{|\zeta|}. \tag{21}$$

One defines the integral operator $G_\zeta : H_{\delta+1}^s(\mathbb{R}^3) \rightarrow H_\delta^s(\mathbb{R}^3)$ by

$$G_\zeta(v) := \mathcal{F}^{-1} \left(\frac{\hat{v}}{|\xi|^2 + 2\zeta \cdot \xi} \right),$$

where \mathcal{F}^{-1} is the inverse Fourier transform. Clearly, for $|\zeta| \geq c > 0$, G_ζ is the inverse operator of $(\Delta + 2\zeta \cdot D)$ and G_ζ is bounded by

$$\|G_\zeta\| \leq \frac{C}{|\zeta|}, \quad (22)$$

for some constant $C = C(\delta, c)$.

As in [5], we choose $\zeta \in \mathbb{C}^3$ such that $\zeta \cdot \zeta = k^2$ and $|\zeta|$ large. Compared to [12, 5], the following proposition extends the construction of CGO solutions to $H_\delta^s(\Omega)$, while requiring more regularities in μ, γ .

Proposition 3.1. *Let $s \geq \frac{3}{2}$ and $-1 < \delta < 0$. Assume that $\mu, \gamma \in H^{s+2}(\Omega)$ and μ, γ are constant outside a compact set. There exists a CGO solution in $H_\delta^s(\Omega)$ of the form*

$$Z = e^{i\zeta \cdot x}(Z_0 + Z_r)$$

to equation (18), such that $Z_0 \in \mathbb{C}^3$ is a constant vector and

$$\|Z_r\|_{H_\delta^s(\Omega)} \leq \mathcal{O}(|\zeta|^{-1}).$$

Proof. Since $\zeta \cdot \zeta = k^2$, we have

$$\begin{aligned} 0 &= e^{-i\zeta \cdot x}(-(\Delta + k^2)I_8 + Q_{\mu, \gamma})e^{i\zeta \cdot x}(Z_0 + Z_r) \\ &= (-\Delta + 2\zeta \cdot D)Z_r + Q_{\mu, \gamma}(Z_0 + Z_r), \end{aligned} \quad (23)$$

or

$$(-\Delta + 2\zeta \cdot D)Z_r + Q_{\mu, \gamma}Z_r = -Q_{\mu, \gamma}Z_0. \quad (24)$$

When $s \geq \frac{3}{2}$, $H^s(\mathbb{R}^3)$ is an algebra. By the assumption, $Q_{\mu, \gamma} \in H^s(\Omega)$ is compactly supported and thus multiplication by $Q_{\mu, \gamma}$ is a bounded operator mapping $H_\delta^s(\mathbb{R}^3)$ to $H_{\delta+1}^s(\mathbb{R}^3)$. In particular, $Q_{\mu, \gamma}Z_0 \in H_{\delta+1}^s(\mathbb{R}^3)$ for any constant Z_0 . The estimate in (22) implies that $I + G_\zeta(Q_{\mu, \gamma} \cdot)$ is well defined and is an invertible mapping $H_\delta^s(\mathbb{R}^3)$ to $H_\delta^s(\mathbb{R}^3)$ for $|\zeta|$ sufficiently large. We can then define

$$Z_r = -(I + G_\zeta(Q_{\mu, \gamma} \cdot))^{-1}G_\zeta(Q_{\mu, \gamma}Z_0) \in H_\delta^s(\mathbb{R}^3). \quad (25)$$

It is immediate that

$$Z_r + G_\zeta(Q_{\mu, \gamma}Z_r) = -G_\zeta(Q_{\mu, \gamma}Z_0). \quad (26)$$

Z_r also satisfies (23) and (24). Thus $e^{i\zeta \cdot x}(Z_0 + Z_r) \in H_\delta^s(\mathbb{R}^3)$ is a solution to (18). We also see from (25) that

$$\|Z_r\|_{H_\delta^s} \leq \mathcal{O}(|\zeta|^{-1})\|Q_{\mu, \gamma}Z_0\|_{H_{\delta+1}^s} \leq \mathcal{O}(|\zeta|^{-1}). \quad (27)$$

□

With CGO solutions to the matrix Schrödinger equation, we can now construct solutions to the original Maxwell equations. We define Y by (17) and thus Y satisfies (16). Note that the solution Y can be written as

$$Y = e^{i\zeta \cdot x}(Y_0 + Y_r), \quad (28)$$

where

$$Y_0 = P^\mp(\zeta)Z_0, \quad Y_r = (P^\mp(\zeta) - W_{\mu,\gamma}^t)Z_r - W_{\mu,\gamma}^t Z_0, \quad (29)$$

satisfying

$$\|Y_0/|\zeta|\|_{L^2} = \mathcal{O}(1), \quad \|Y_r/|\zeta|\|_{H_\delta^{s-1}} = \mathcal{O}(|\zeta|^{-1}). \quad (30)$$

Here, Y_0 is a constant vector depending on the choice of ζ and Z_0 . We denote the components of Y by

$$Y = (Y^\Phi, (Y^H)^t, Y^\Psi, (Y^E)^t)^t.$$

We recall that (16) is equivalent to the original Maxwell equations only if $Y^\Phi = Y^\Psi = 0$. This condition can be satisfied with properly chosen Z_0 according to the following lemma, which is cited from [5] without proof.

Lemma 3.2 ([5]). *If*

$$((P^\mp(\zeta) - k)Z_0)^\Phi = ((P^\mp(\zeta) - k)Z_0)^\Psi = 0,$$

then we have

$$Y^t = (0 \ (Y^H)^t \ 0 \ (Y^E)^t)$$

for $|\zeta|$ sufficiently large.

Proposition 3.3. *The Maxwell equations have a solution of the form*

$$X = e^{i\zeta \cdot x}(X_0 + X_r)$$

satisfying

$$\|X_0/|\zeta|\|_{H_\delta^{s+2}} = \mathcal{O}(1), \quad \|X_1/|\zeta|\|_{H_\delta^{s-1}} = \mathcal{O}(|\zeta|^{-1}).$$

The proof of the above proposition follows directly from (15) and Lemma 3.2. An explicit choice of Z_0 satisfying Lemma 3.2 is given by

$$Z_0 = \frac{1}{|\zeta|} \begin{pmatrix} \zeta \cdot a \\ kb \\ \zeta \cdot b \\ ka \end{pmatrix},$$

where a, b are any vectors in \mathbb{C}^3 . It follows that

$$X_0 = \text{diag}(\mu^{-1/2}, \gamma^{-1/2})P^\mp(\zeta)Z_0 = \frac{1}{|\zeta|} \begin{pmatrix} ka \cdot \zeta / \mu^{-1/2} \\ ((b \cdot \zeta)\zeta - k\zeta \times a) / \mu^{-1/2} \\ kb \cdot \zeta / \gamma^{-1/2} \\ ((a \cdot \zeta)\zeta + k\zeta \times b) / \gamma^{-1/2} \end{pmatrix}.$$

We choose $b \in \mathbb{C}^3$ such that $|b| = \mathcal{O}(|\zeta|)$, and define a CGO solution as

$$\check{E} = \frac{1}{|\zeta|} X^E = e^{i\zeta \cdot x}(\check{E}_0 + \check{E}_r), \quad (31)$$

where

$$\check{E}_0 = \frac{\sqrt{\gamma}}{|\zeta|^2}((a \cdot \zeta)\zeta + k\zeta \times b) \in H^{s+2}(\Omega) \quad (32)$$

and

$$\|\check{E}_0\|_{H^{s+2}(\Omega)} = \mathcal{O}(1), \quad \|\check{E}_r\|_{H^{s-1}(\Omega)} = \mathcal{O}(|\zeta|^{-1}). \quad (33)$$

Compared to the CGO solutions in [7], our new construction applies to variable μ , while the required regularity of parameters μ, γ is one order higher. Here and below, we use a check sign, for example \check{E} , to indicate corresponding variables computed from the CGO solutions.

4 Inversion of Maxwell's equations with internal data

In this section, we state and prove the uniqueness and stability results of the inverse problem of Maxwell's equations with internal data. In addition to Sobolev spaces $H^s(\Omega)$, we also work on continuous function spaces $C^d(\Omega)$. Here s and d are related by $s = \frac{3}{2} + d + 2 + \iota$ for some small $\iota > 0$. We assume that $\mu \in H^{\frac{3}{2}+d+4+\iota}(\Omega)$ is known and define \mathcal{M} to be the set of coefficients as

$$\mathcal{M} := \{(L, \gamma) \in C^{d+1}(\overline{\Omega}) \times H^{\frac{3}{2}+d+4+\iota}(\Omega) : d \geq 3, \iota > 0 \text{ is small, and } 0 \text{ is not an eigenvalue of } D \wedge (\mu^{-1} D \wedge \cdot) + \omega^2 \gamma\}. \quad (34)$$

Our proof makes use of CGO solutions, constructed in previous section, of the form (31). The trace of the CGO solutions in $H^{d+3+\iota}(\partial\Omega)$ describes how we should control the boundary values of electric fields. We note that the dominant term, $e^{i\zeta \cdot x} \check{E}_0$, of a CGO solution is characterized by parameters ζ, a, b and the coefficient γ . Therefore, for a particular choice of ζ, a and b and γ known on the boundary of the domain, the restriction of $e^{i\zeta \cdot x} \check{E}_0$ to the boundary gives an explicit form of the source distribution. In our inversion method, we construct two CGO solutions, according to which we define two electric sources, denoted by (G_1, G_2) . The following theorems state that, with electric sources (G_1, G_2) , we can invert Λ_G uniquely and stably.

Theorem 4.1. *Let $d \geq 3$. Let Ω be an open bounded subset of \mathbb{R}^3 with boundary $\partial\Omega$ of class C^d . Let (L, γ) and $(\tilde{L}, \tilde{\gamma})$ be two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. Let $\Sigma := (\Sigma_1, \Sigma_2)$ and $\tilde{\Sigma} := (\tilde{\Sigma}_1, \tilde{\Sigma}_2)$ be two sets of internal data on Ω for coefficients (L, γ) and $(\tilde{L}, \tilde{\gamma})$, respectively, and with boundary values $G := (G_1, G_2)$.*

There is a non-empty open set of G in $(H^{d+3+\iota}(\partial\Omega))^2$, defined as a neighborhood of the trace of CGO solutions in (31), for small $\iota > 0$, such that if $\Sigma_j = \tilde{\Sigma}_j$ in $C^{d+1}(\overline{\Omega})$, $j = 1, 2$, we have $(L, \gamma) = (\tilde{L}, \tilde{\gamma})$.

From the proof it will become apparent that the implication $L = \tilde{L}$ requires that $d \geq 1$ and that the implication $\gamma = \tilde{\gamma}$ requires that $d \geq 3$. Here and in the following, we shall abuse the notation and use $C^d(\overline{\Omega})$ to denote either set of complex-valued functions or set of vector-valued functions the elements of which have up to d -th order continuous derivatives. The function space $(H^{d+3+\iota}(\partial\Omega))^2$ is an abbreviation of the product space $H^{d+3+\iota}(\partial\Omega) \times H^{d+3+\iota}(\partial\Omega)$.

The proof of the uniqueness theorem establishes an explicit reconstruction. Following this reconstruction we also prove a Lipschitz stability result. To consider the stability of the reconstruction, we need to restrict it to a subset of Ω . Let ζ_0 be a constant unit vector close to $\zeta_1/|\zeta_1|$, where ζ_1 occurs in the CGO solution. Explicit

choice of ζ_0 and ζ_1 is given in (41) and (42). Define Ω_1 to be the subset of Ω obtained by removing a neighborhood of every point $x_0 \in \partial\Omega$ such that $n(x_0) \cdot \zeta_0 = 0$, where $n(x_0)$ is the outward normal of $\partial\Omega$ at x_0 .

Theorem 4.2. *Let $d \geq 3$. Let Ω be convex with boundary $\partial\Omega$ of class C^d and Ω_1 be defined as above. Let (L, γ) and $(\tilde{L}, \tilde{\gamma})$ be two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. Let $\Sigma := (\Sigma_1, \Sigma_2)$ and $\tilde{\Sigma} := (\tilde{\Sigma}_1, \tilde{\Sigma}_2)$ be two sets of internal data on Ω for coefficients (L, γ) and $(\tilde{L}, \tilde{\gamma})$, respectively, and with boundary values $G := (G_1, G_2)$.*

There is a non-empty open set of G in $(H^{d+3+\iota}(\partial\Omega))^2$, defined as a neighborhood of the trace of CGO solutions in (31), for small $\iota > 0$ such that

$$\|L - \tilde{L}\|_{C^{d-1}(\overline{\Omega}_1)} + \|\gamma - \tilde{\gamma}\|_{C^{d-3}(\overline{\Omega}_1)} \leq C\|\Sigma - \tilde{\Sigma}\|_{(C^{d+1}(\overline{\Omega}_1))^2}, \quad (35)$$

for some constant C .

We note that the constant C is in the order of $\mathcal{O}(|\zeta_1|^{-1}(\|\Sigma\|_{C^{d+1}} + \|\tilde{\Sigma}\|_{C^{d+1}}))$, which indicates large $|\zeta_1|$ leading to better estimate.

If we have at least 6 sets of complex measurements, we can prove the same stability results with the subset requirement removed.

Theorem 4.3. *Let $d \geq 3$. Let Ω be convex with boundary $\partial\Omega$ of class C^d . Let (L, γ) and $(\tilde{L}, \tilde{\gamma})$ be two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. For $j = 1, 2, 3$, let $\Sigma := (\Sigma_1^j, \Sigma_2^j)$ and $\tilde{\Sigma} := (\tilde{\Sigma}_1^j, \tilde{\Sigma}_2^j)$ be two sets of internal data on Ω for coefficients (L, γ) and $(\tilde{L}, \tilde{\gamma})$, respectively, and with boundary values $G := (G_1^j, G_2^j)$.*

Then there is a non-empty open set of G in $(H^{d+3+\iota}(\partial\Omega))^6$, defined as a neighborhood of the trace of CGO solutions in (31), for small $\iota > 0$, such that

$$\|L - \tilde{L}\|_{C^{d-1}(\overline{\Omega})} + \|\gamma - \tilde{\gamma}\|_{C^{d-3}(\overline{\Omega})} \leq C\|\Sigma - \tilde{\Sigma}\|_{(C^{d+1}(\overline{\Omega}))^6}, \quad (36)$$

for some constant C .

Our proof relies on the unique solvability of a transport equation, the vector field of which can be controlled by properly constructed CGO solutions of Maxwell's equations.

4.1 Construction of vector fields and uniqueness result

We denote the curl-curl form of the Maxwell's equation by

$$F[E] := \nabla \times (\mu^{-1} \nabla \times E) - \omega^2 \gamma E = 0.$$

We let E_1, E_2 be two solutions to this equation, then

$$\begin{aligned} 0 &= F[E_1] \cdot E_2 - F[E_2] \cdot E_1 \\ &= \nabla \mu^{-1} \times \nabla \times E_1 \cdot E_2 - \nabla \mu^{-1} \times \nabla \times E_2 \cdot E_1 \\ &\quad + \mu^{-1} (\nabla \times \nabla \times E_1 \cdot E_2 - \nabla \times \nabla \times E_2 \cdot E_1). \end{aligned} \quad (37)$$

We recall that $E = \Sigma/L$. By some calculations, we get

$$\phi \cdot \nabla L + \psi L = 0, \quad (38)$$

where

$$\begin{aligned} \phi &= \chi(x)[[(\nabla\mu^{-1} \cdot \Sigma_1)\Sigma_2 - (\nabla\mu^{-1} \cdot \Sigma_2)\Sigma_1] + [(\nabla\Sigma_1)\Sigma_2 - (\nabla\Sigma_2)\Sigma_1] \\ &\quad [(\nabla \cdot \Sigma_1)\Sigma_2 - (\nabla \cdot \Sigma_2)\Sigma_1] - 2[(\nabla\Sigma_1)^T\Sigma_2 - (\nabla\Sigma_2)^T\Sigma_1]], \end{aligned} \quad (39)$$

$$\begin{aligned} \psi &= \chi(x)[\nabla\mu^{-1}[(\nabla \times \Sigma_1) \times \Sigma_2 - (\nabla \times \Sigma_2) \times \Sigma_1] \\ &\quad + [\nabla(\nabla \cdot \Sigma_1)\Sigma_2 - \nabla(\nabla \cdot \Sigma_2)\Sigma_1] - [\nabla^2\Sigma_1 \cdot \Sigma_2 - \nabla^2\Sigma_2 \cdot \Sigma_1]]. \end{aligned} \quad (40)$$

Here $\chi(x)$ is any smooth nonzero complex-valued function.

By the method of characteristics, the transport equation (38) has a unique solution only if the integral curves of ϕ connect every point in Ω to a point on $\partial\Omega$. We will prove that, with properly chosen CGO solutions, the integral curves of ϕ will be close to straight lines, and thus connect every internal point to two boundary points.

We make specific choices for ζ, a, b in (31):

$$\begin{aligned} \zeta_1 &= (1/2, i\sqrt{1/h^2 + 1/4 - k^2}, 1/h), \\ \zeta_2 &= (1/2, -i\sqrt{1/h^2 + 1/4 - k^2}, -1/h), \end{aligned} \quad (41)$$

with $h > 0$ being a free parameter such that $h|\zeta_j| \approx 1$ for $j = 1, 2$. We note that

$$\lim_{h \rightarrow 0} \zeta_1/|\zeta_1| = \frac{1}{\sqrt{2}}(0, i, 1) =: \zeta_0, \quad (42)$$

$$\lim_{h \rightarrow 0} \zeta_2/|\zeta_2| = -\zeta_0 \quad (43)$$

and

$$\zeta_1 + \zeta_2 = (1, 0, 0), \quad \zeta_0 \cdot \zeta_0 = 0.$$

By choosing any $a_j \in \mathbb{C}^3$ and $b_1 = b_2 = (0, 0, 1/h)$, we can verify that CGO solutions \check{E}_1 and \check{E}_2 defined in (31) satisfy

$$\begin{aligned} (\check{E}_0^1 + \check{E}_r^1) \cdot (\check{E}_0^2 + \check{E}_r^2) &= \check{E}_0^1 \cdot \check{E}_0^2 + \mathcal{O}(h) \\ &= \frac{k^2}{|\zeta_1|^2 |\zeta_2|^2 \gamma} (\zeta_1 \times b) \cdot (\zeta_2 \times b) + \mathcal{O}(h) \\ &= \frac{k^2}{|\zeta_1|^2 |\zeta_2|^2 \gamma} \left(\frac{1}{h^2} + \frac{1}{2} - k^2 \right) \frac{1}{h^2} + \mathcal{O}(h) \\ &= \frac{k^2}{\gamma} \frac{1}{|\zeta_1|^2 |\zeta_2|^2 h^4} + \mathcal{O}(h) \end{aligned}$$

and

$$\frac{\zeta_i}{|\zeta_i|} \cdot \check{E}_j = \mathcal{O}(h), \quad \text{for } i, j = 1, 2.$$

We recall that the check sign, for example \check{E}_j , indicates fields corresponding with the CGO solutions. We denote $\check{\Sigma}_j = L\check{E}_j = \vartheta_j e^{i\zeta \cdot x}(\eta_j + R_j)$ with $\vartheta = \frac{Lk}{\gamma^{1/2}}$, $\eta_j = \gamma^{1/2}\check{E}_0^j/k$ and $R = \gamma^{1/2}\check{E}_R/k$. Then $\eta_1 \cdot \eta_2 = 1 + \mathcal{O}(h)$.

We now evaluate $\check{\phi}$. We find that

$$\nabla \check{\Sigma}_j = e^{i\zeta \cdot x}[(\eta_j + R_j)(\nabla \vartheta)^T + i\vartheta(\eta_j + R_j)\zeta^T + \vartheta \nabla(\eta_j + R_j)]$$

and

$$\begin{aligned}
& e^{-i(\zeta_1+\zeta_2)\cdot x} \frac{h}{\vartheta} (\nabla \check{\Sigma}_1) \check{\Sigma}_2 = \nabla(\vartheta E_1) E_2 \\
& = h[(\eta_1 + R_1)(\nabla \vartheta)^T + i\vartheta(\eta_1 + R_1)\zeta_1^T + \vartheta \nabla(\eta_1 + R_1)](\eta_2 + R_2) \\
& = h[(\eta_1 + R_1)[\nabla \vartheta^T(\eta_2 + R_2) + i\vartheta \zeta_1^T(\eta_2 + R_2)] \\
& \quad + \vartheta \nabla(\eta_1 + R_1)(\eta_2 + R_2)] \\
& = i\vartheta h(\zeta_1 \cdot \eta_2) \eta_1 + \mathcal{O}(h) \\
& \rightarrow 0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

Similarly, on a bounded domain,

$$\begin{aligned}
& e^{-i(\zeta_1+\zeta_2)\cdot x} \frac{h}{\vartheta} (\nabla \check{\Sigma}_1)^T \check{\Sigma}_2 \\
& = h[\nabla \vartheta(\eta_1 + R_1)^T(\eta_2 + R_2) + i\vartheta \zeta_1(\eta_1 + R_1)^T(\eta_2 + R_2) \\
& \quad + \vartheta \nabla(\eta_1 + R_1)^T(\eta_2 + R_2)] \\
& = i\vartheta h \zeta_1 + \mathcal{O}(h) \\
& \rightarrow i\vartheta \zeta_0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& e^{-i(\zeta_1+\zeta_2)\cdot x} \frac{h}{\vartheta} (\nabla \cdot \check{\Sigma}_1) \check{\Sigma}_2 \\
& = h[(\nabla \vartheta \cdot (\eta_1 + R_1))(\eta_2 + R_2) + i\vartheta(\zeta_1 \cdot (\eta_1 + R_1))(\eta_2 + R_2) \\
& \quad + \vartheta \nabla \cdot (\eta_1 + R_1)(\eta_2 + R_2)] \\
& = i\vartheta h(\zeta_1 \cdot \eta_1) \eta_2 + \mathcal{O}(h) \\
& \rightarrow 0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

We also find that

$$\begin{aligned}
& e^{-i(\zeta_1+\zeta_2)\cdot x} \frac{h}{\vartheta} (\nabla \frac{1}{\mu} \cdot \check{\Sigma}_1) \check{\Sigma}_2 \\
& = \vartheta h (\nabla \frac{1}{\mu} \cdot \eta_1) \eta_2 + \mathcal{O}(h) \\
& \rightarrow 0 \quad \text{as } h \rightarrow 0.
\end{aligned}$$

We choose $\chi(x) = -\frac{h}{4} e^{-i(\zeta_1+\zeta_2)\cdot x}$; then

$$\|\check{\phi} - i\vartheta^2 \zeta_0\|_{H_{\delta}^{s-1}(\mathbb{R}^3)} = \mathcal{O}(h). \quad (44)$$

We let $s = \frac{3}{2} + d + 2 + \iota$ for $d > 0$ and small $\iota > 0$. The Sobolev embedding theorem implies that

$$\|\check{\phi} - i\vartheta^2 \zeta_0\|_{C_{\delta}^{d+1}(\mathbb{R}^3)} = \mathcal{O}(h). \quad (45)$$

If we choose $b_1 = b_2 = (0, 0, h^{\varepsilon-1})$ and $\chi(x) = -\frac{h^{1-2\varepsilon}}{4} e^{-i(\zeta_1+\zeta_2)\cdot x}$ with $0 < \varepsilon < 1/2$, the same calculations as above give

$$\|\check{\phi} - i\vartheta^2 \zeta_0\|_{C_{\delta}^{d+1}(\mathbb{R}^3)} = \mathcal{O}(h^{1-2\varepsilon}). \quad (46)$$

So far we proved that if the electric fields are exactly given by the CGO solutions as we constructed, the vector field $\check{\phi}$ in (38) can be approximated by (44), which implies that every integral curve of $\check{\phi}$ is almost a straight line and thus connects every internal point to two boundary points. Next, we prove that we can perturb the CGO solutions so that an estimate similar to (44) still holds. To follow the dependencies of vector fields on boundary conditions, we introduce a regularity theorem for Maxwell's equations. Let tE be the tangential boundary condition of E . The following function spaces were introduced in [11]:

$$H_{\text{Div}}^l(\Omega) = \{u \in H^l\Omega^1(\Omega) : \text{Div}(tu) \in H^{l-1/2}(\partial\Omega)\},$$

$$TH_{\text{Div}}^l(\partial\Omega) = \{g \in H^l\Omega^1(\partial\Omega) : \text{Div}(g) \in H^l(\partial\Omega)\},$$

for $l > 0$, where $H^l\Omega^1(\Omega)$ is a space of vector functions each component of which is in $H^l(\Omega)$. These are Hilbert spaces with norms

$$\|u\|_{H_{\text{Div}}^l(\Omega)} = \|u\|_{H^l(\Omega)} + \|\text{Div}(tu)\|_{H^{l-1/2}(\partial\Omega)},$$

$$\|g\|_{TH_{\text{Div}}^l(\partial\Omega)} = \|g\|_{H^l(\partial\Omega)} + \|\text{Div}(g)\|_{H^l(\partial\Omega)}.$$

It is clear that $t(H_{\text{Div}}^l(\Omega)) = TH_{\text{Div}}^{l-1/2}(\partial\Omega)$. We also observe that

$$\|E\|_{H^l(\Omega)} \leq \|E\|_{H_{\text{Div}}^l(\Omega)} \text{ and } \|G\|_{TH_{\text{Div}}^l(\partial\Omega)} \leq \|G\|_{H^{l+1}(\partial\Omega)} \quad (47)$$

Proposition 4.4 ([11]). *Let $\epsilon, \mu \in C^l$, $l > 2$, be positive functions. There is a discrete subset $\Sigma \subset \mathbb{C}$, such that if $\omega \notin \Sigma$, then one has a solution $E \in H_{\text{Div}}^l$ to (7) given any tangential boundary condition $G \in TH_{\text{Div}}^{l-1/2}(\partial\Omega)$. The solution satisfies*

$$\|E\|_{H_{\text{Div}}^l(\Omega)} \leq C\|G\|_{H_{\text{Div}}^{l-1/2}} \quad (48)$$

with C independent of G .

We now estimate vector field ϕ in (38) when the boundary value of the electric field is given by a perturbation of the trace of a CGO solution.

Proposition 4.5. *Under the assumptions of Proposition 4.4, when G_j is in a neighborhood of $\tilde{G}_j = t\tilde{E}_j \in H^{d+3+\iota}(\partial\Omega)$, $j = 1, 2$, the corresponding vector field ϕ defined in (39) satisfies*

$$\|\phi - i\vartheta^2\zeta_0\|_{C^d(\overline{\Omega})} = \mathcal{O}(h) \quad (49)$$

for small h .

Let $s = \frac{3}{2} + d + 2 + \iota$, for small $\iota > 0$. Then this proposition follows from Proposition 3.6 in [7] directly. To be self-contained in this section, we summarize the proof here.

Proof. According to the Sobolev embedding theorem, proposition 4.4 and eq. (47), we have that

$$\begin{aligned} \|E\|_{C^{d+1}(\overline{\Omega})} &\leq C\|E\|_{H^{\frac{5}{2}+d+\iota}(\Omega)} \leq C\|E\|_{H_{\text{Div}}^{\frac{5}{2}+d+\iota}(\Omega)} \\ &\leq C\|G\|_{TH_{\text{Div}}^{d+2+\iota}(\partial\Omega)} \leq C\|G\|_{H^{d+3+\iota}(\partial\Omega)}, \end{aligned}$$

where various constants are all named “C”. Hence

$$\|E\|_{C^{d+1}(\bar{\Omega})} \leq C\|G\|_{H^{d+3+\iota}(\partial\Omega)}. \quad (50)$$

Let us now define boundary conditions $G_j \in H^{d+3+\iota}(\partial\Omega)$, $j = 1, 2$, such that

$$\|G_j - t\check{E}_j\|_{H^{d+3+\iota}(\partial\Omega)} \leq \varepsilon, \quad (51)$$

for some $\varepsilon > 0$ sufficiently small. let E_j be the solution to the Maxwell equations (9) with $tE_j = G_j$. By (50), we thus have

$$\|E_j - \check{E}_j\|_{C^{d+1}(\bar{\Omega})} \leq C\varepsilon, \quad (52)$$

for some positive constant C . We introduce the complex-valued internal data, $\Sigma_j = LE_j$ and conclude that

$$\|\Sigma_j - \check{\Sigma}_j\|_{C^{d+1}(\bar{\Omega})} \leq C\varepsilon. \quad (53)$$

We obtain the estimate (cf. (39))

$$\|\phi - \check{\phi}\|_{C^d(\bar{\Omega})} \leq C\|\chi(x)\|_{C^d(\bar{\Omega})}\|\Sigma_j - \check{\Sigma}_j\|_{C^{d+1}(\bar{\Omega})} \leq Ch\varepsilon, \quad (54)$$

where $\chi(x)$ is defined above (44). Therefore, (49) follows from (45) and (54). This completes the proof. \square

We recall that \mathcal{M} is the parameter space of (L, γ) defined in (34). We now prove Theorem 4.1.

Proof of Theorem 4.1. Let $d \geq 3$. By proposition 4.5, we choose the set of boundary conditions for electrical fields to be a neighborhood of $(\check{G}_j) = (t\check{E}_j)$ in $H^{d+3+\iota}(\partial\Omega)^2$. By assuming that the measurements coincide, that is, $\Sigma = \check{\Sigma}$, we have that $\phi = \check{\phi}$ and $\psi = \check{\psi}$ by (39)-(40). Thus, L and \check{L} solve the same transport equation (38) while $L = \check{L} = \Sigma/G$ on $\partial\Omega$. As ϕ satisfies (49), we deduce that $L = \check{L}$ since integral curves of ϕ map any $x \in \Omega$ to two boundary points. More precisely, consider the flow $\theta_x(t)$ associated with the imaginary part of ϕ , that is, $\theta_x(t)$ is the solution to

$$\dot{\theta}_x(t) = \Im\phi(\theta_x(t)), \quad \theta(0) = x \in \bar{\Omega}.$$

By the Picard-Lindelöf theorem, (55) admits a unique solution since ϕ is of class $C^1(\Omega)$. Also by (49), for $\forall x \in \Omega$, there exist $x_{\pm}(x) \in \partial\Omega$ and $t_{\pm}(x) > 0$ such that

$$\theta_x(t_{\pm}(x)) = x_{\pm}(x) \in \partial\Omega.$$

By the method of characteristics, the solution L to the transport equation (38) is given by

$$L(x) = L_0(x_{\pm}(x))e^{-\int_0^{t_{\pm}(x)} \Im\gamma(\theta_x(s))ds}, \quad (55)$$

where $L_0 := L|_{\partial\Omega}$ is the restriction of L on the boundary. The solution \check{L} is given by the same formula since $\theta_x(t) = \check{\theta}_x(t)$. This implies that $L = \check{L}$ and thus $E_j = \check{E}_j = \Sigma_j/L = \check{\Sigma}_j/\check{L}$, $j = 1, 2$. By the choice of illuminations, we also have $|E_j| \neq 0$ due to (52) and $|\check{E}_j| \neq 0$. Finally, by substituting E_j and \check{E}_j into (9), we can solve for γ and $\tilde{\gamma}$ by

$$\gamma = -\frac{(\Sigma \wedge \frac{1}{\mu}\Sigma \wedge E) \cdot \bar{E}}{\omega^2|E|^2} \quad \text{and} \quad \tilde{\gamma} = -\frac{(\Sigma \wedge \frac{1}{\mu}\Sigma \wedge \tilde{E}) \cdot \tilde{\bar{E}}}{\omega^2|\tilde{E}|^2}. \quad (56)$$

We then conclude that $\gamma = \tilde{\gamma}$ in $C^{d-3}(\Omega)$ for $d \geq 3$. \square

4.2 Stability result

The proof of the stability theorem follows the argument of [7] with the estimate of the vector field replaced by (49).

Proposition 4.6. *Let $d \geq 1$. Let L and \tilde{L} be solutions to (38) corresponding to coefficients (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$, respectively, where (49) holds for both ϕ and $\tilde{\phi}$.*

Let $L_0 = L|_{\partial\Omega}$, $\tilde{L}_0 = \tilde{L}|_{\partial\Omega}$, and $L_0, \tilde{L}_0 \in C^{d-1}(\partial\Omega)$. We also assume that h is sufficiently small and that Ω is convex. Then there is a constant C such that upon restricting to Ω_1 ,

$$\begin{aligned} \|L - \tilde{L}\|_{C^{d-1}(\overline{\Omega}_1)} &\leq C \|L_0\|_{C^{d-1}(\partial\Omega_1)} [\|\phi - \tilde{\phi}\|_{C^d(\overline{\Omega}_1)} \\ &\quad + \|\psi - \tilde{\psi}\|_{C^{d-1}(\overline{\Omega}_1)}] + C \|L_0 - \tilde{L}_0\|_{C^{d-1}(\partial\Omega_1)}. \end{aligned} \quad (57)$$

The choice of Ω_1 depends on the proof of the above proposition. Readers are referred to [7] for the details. Now we can prove the main stability theorem.

Proof of Theorem 4.2. From (39) and (40), it is straightforward to check that

$$\begin{aligned} \|\phi - \tilde{\phi}\|_{C^d(\overline{\Omega}_1)} &\leq C \|\Sigma - \tilde{\Sigma}\|_{C^{d+1}(\overline{\Omega}_1)} \text{ and} \\ \|\psi - \tilde{\psi}\|_{C^{d-1}(\overline{\Omega}_1)} &\leq C \|\Sigma - \tilde{\Sigma}\|_{C^{d+1}(\overline{\Omega}_1)}, \end{aligned}$$

where $C > 0$ is a positive constant. The first part of (35) then follows directly from Proposition 4.6. To estimate the difference between γ and $\tilde{\gamma}$, we notice that

$$E - \tilde{E} = \frac{\Sigma}{L} - \frac{\tilde{\Sigma}}{\tilde{L}} = \frac{L(\Sigma - \tilde{\Sigma}) - \Sigma(L - \tilde{L})}{L\tilde{L}}.$$

Since L and \tilde{L} are non-vanishing, by the stability result for L we obtain

$$\|E - \tilde{E}\|_{C^{d-1}(\overline{\Omega}_1)} \leq C \|\Sigma - \tilde{\Sigma}\|_{C^{d+1}(\overline{\Omega}_1)}. \quad (58)$$

By choosing the boundary values close to the boundary conditions of CGO solutions, (51) and (52) imply that E_j is non-vanishing since the CGO solutions are non-vanishing. We recall that γ and $\tilde{\gamma}$ are computed by (56). By taking the difference and using (58) we derive

$$\|\gamma - \tilde{\gamma}\|_{C^{d-3}(\overline{\Omega}_1)} \leq C \|\Sigma - \tilde{\Sigma}\|_{C^{d+1}(\overline{\Omega}_1)}.$$

This completes the proof. \square

The required regularity of the different variables is summarized in Table 1, where $s = \frac{3}{2} + d + 2 + \iota$.

4.3 Stability with 6 complex internal data

We now follow Section 3.4 of [7] to prove Theorem 4.3. If we take more internal measurements, we can rewrite (38) into matrix form. We first construct proper CGO solutions. Let $j = 1, 2, 3$ in this section. We can choose unit vectors ζ_0^j , such

Table 1: Regularity of fields and coefficients

μ, γ	$H^{s+2}(\mathbb{R}^3)$	Required regularity for the construction of CGO's
$Q_{\mu, \gamma}, Z$	$H^s(\mathbb{R}^3)$	$s \geq \frac{3}{2}$ and $Q_{\mu, \gamma}$ is compactly supported
$Y, X, \tilde{E}, \tilde{\Sigma}$	$H^{s-1}(\mathbb{R}^3)$	Constructed regularity
G	$H^{s-\frac{1}{2}}(\partial\Omega)$	Required regularity for boundary source
L, Σ	$H^{s-1}(\overline{\Omega})$	Required regularity for inversion, $H^{s-1}(\overline{\Omega}) \subset C^{d+1}(\overline{\Omega})$
ϕ	$C^d(\overline{\Omega})$	
ψ, L, E	$C^{d-1}(\overline{\Omega})$	Recovered regularity, $d \geq 1$
γ	$C^{d-3}(\overline{\Omega})$	Recovered regularity, $d \geq 3$

that $\zeta_0^j \cdot \zeta_0^j = 0$ and $\{\zeta_0^j\}$ are linearly independent. We next choose (ζ_1^j, ζ_2^j) such that $|\zeta| := |\zeta_1^j| = |\zeta_2^j|$ and

$$\lim_{|\zeta| \rightarrow \infty} \frac{\zeta_1^j}{|\zeta|} = \lim_{|\zeta| \rightarrow \infty} \frac{\zeta_2^j}{|\zeta|} = \zeta_0^j.$$

(a_1^j, a_2^j) and (b_1^j, b_2^j) are chosen such that $a_1^j = a_2^j$, $b_1^j = b_2^j$,

$$a_1^j \cdot \zeta_0^j = 0 \quad \text{and} \quad 0 < |\zeta_1^j \times b_1^j| = \mathcal{O}(|\zeta_1^j|^2).$$

We construct CGO solutions $\tilde{E}_1^j, \tilde{E}_2^j$ corresponding to $(\zeta_1^j, a_1^j, b_1^j)$ and $(\zeta_2^j, a_2^j, b_2^j)$, for $j = 1, 2, 3$. Let the boundary illuminations G_1^j, G_2^j be chosen according to (51). The measured internal data are then given by D_1^j, D_2^j . Proposition 4.4 shows that the vector field defined by (39) satisfies

$$\|\phi^j - i\vartheta^2 \zeta_0^j\|_{C^d(\overline{\Omega})} \leq \frac{C}{|\zeta|}.$$

The rest of the proof of Theorem 4.3 then follows the argument in Section 3.4 of [7].

5 The temporal behavior of CGO solutions

Here, we characterize the internal data following from the CGO solutions introduced above, in particular, in the small h limit. We find that their frequency behavior is approximately Gaussian.

Let ζ_1 and ζ_2 be defined in (41). We notice that, for small h ,

$$\sqrt{\frac{1}{h^2} + \frac{1}{4} - k^2} = \sqrt{\frac{1}{h^2} + \frac{1}{4}} - \frac{hk^2}{2\sqrt{1 + \frac{h^2}{4}}} + \mathcal{O}(h^3 k^4). \quad (59)$$

We let

$$\begin{aligned} \tilde{\zeta}_1 &= \left(\frac{1}{2}, \frac{1}{h}, i\sqrt{\frac{1}{h^2} + \frac{1}{4}} \right) \\ \tilde{\zeta}_2 &= \left(\frac{1}{2}, -\frac{1}{h}, -i\sqrt{\frac{1}{h^2} + \frac{1}{4}} \right). \end{aligned}$$

To generate CGO solutions with Gaussian behavior in frequency, we consider two choices of parameters (a_j, b_j) , $j = 1, 2$:

(1) We choose $a_j \in \mathbb{C}^3$ such that $a_j \cdot \zeta_j = 0$ for $j = 1, 2$, and $b_1 = b_2 = (0, 0, 1/h)$. Then $\check{E}_{0,j} = i\omega \tilde{E}_{0,j} + \mathcal{O}(h^2\omega^2)$, where $\tilde{E}_{0,j}$ is independent of ω ,

$$\tilde{E}_{0,j} = \frac{\mu_0\gamma_0\sqrt{\gamma}}{i|\zeta|^2}(\tilde{\zeta}_j \times b) = i\frac{\mu_0\gamma_0\sqrt{\gamma}}{|\zeta|^2h^2} \left((-1)^j, \frac{h}{2}, 0 \right).$$

Assuming that ω is bounded, we have

$$\begin{aligned} \check{E}_1(x, \omega) &= e^{-\alpha h\omega^2} e^{i\tilde{\zeta}_1 \cdot x} (i\omega \tilde{E}_{0,1} + \mathcal{O}(h)), \\ \check{E}_2(x, \omega) &= e^{\alpha h\omega^2} e^{i\tilde{\zeta}_2 \cdot x} (i\omega \tilde{E}_{0,2} + \mathcal{O}(h)), \end{aligned} \tag{60}$$

where

$$\alpha = \frac{\mu_0\gamma_0 x_3}{2\sqrt{1 + \frac{h^2}{4}}}.$$

(2) We choose a_j such that $a_j \cdot \zeta_j = 1/h$ and $b_1 = b_2 = (0, 0, h^{\varepsilon-1})$ for $0 < \varepsilon < 1/2$. Then $E_{0,j} = \tilde{E}_{0,j} + \mathcal{O}(i\omega h^\varepsilon)$, where

$$\tilde{E}_{0,j} = \frac{\sqrt{\gamma}}{|\zeta_j|^2 h} \tilde{\zeta}_j.$$

Then the CGO solutions are

$$\begin{aligned} \check{E}_1(x, \omega) &= e^{-\alpha h\omega^2} e^{i\tilde{\zeta}_1 \cdot x} (\tilde{E}_{0,1}(x) + \mathcal{O}(i\omega h^\varepsilon)), \\ \check{E}_2(x, \omega) &= e^{\alpha h\omega^2} e^{i\tilde{\zeta}_2 \cdot x} (\tilde{E}_{0,2}(x) + \mathcal{O}(i\omega h^\varepsilon)). \end{aligned} \tag{61}$$

Remark: Although the choice of b_j such that $\zeta_j \times b_j = 0$ could simplify the above argument, it will destroy the proof in Section 4.1. Also, in the second choice of (a_j, b_j) , we would prefer large ε to get better approximations in (61). However, the proof in Section 4.1 requires larger $1 - 2\varepsilon$ in (46), that is, smaller ε , to obtain a better stability estimate. We need to choose ε to balance these two conditions.

Suppose ω is bounded. Upon taking an inverse Fourier transform of (60) and (61), we obtain

$$\mathcal{F}^{-1} \check{E}_j(x, t) \approx f(t) e^{i\tilde{\zeta}_j \cdot x} \tilde{E}_{0,j}(x), \quad \text{for } j = 1, 2,$$

where $f(t)$ is a function in time and concentrates at zero time while h is small. The above equation also characterizes the temporal behavior of electric source G and the internal data Σ . In experiments, one can generate an electric source with particular waveform by convolving G with a window function, \hat{W} say, centered at a particular frequency corresponding with the certain waveform in time.

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